NOTE ON THE BLOWUP CRITERION OF SMOOTH SOLUTION TO THE INCOMPRESSIBLE VISCOELASTIC FLOW

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Abstract. We study the blowup criterion of smooth solution to the Oldroyd models. Let $(u(t,x), F(t,x)$ be a smooth solution in $[0,T)$, it is shown that the solution $(u(t,x), F(t,x)$ does not appear breakdown until $t = T$ provided $\nabla u(t,x) \in L^1([0,T]; L^\infty(\mathbb{R}^n))$, $n = 2, 3$.

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1. Introduction

In this paper, we consider the blowup criterion of smooth solution to the incompressible Oldroy model in the two and three dimensional space:

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = \nabla\cdot(FF^t), \\ \partial_t F + u\cdot\nabla F = \nabla u F, \\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

for any $t > 0$, $x \in \mathbb{R}^n$, $n = 2,\ 3$, where $u(t,x)$ is the velocity field, p is the pressure, μ is the viscosity and F the deformation tensor. We denote $(\nabla\cdot F)_i = \partial_{x_j} F_{ij}$ for a matrix F. The Oldroy model (1.1) describes an incompressible non-Newtonian fluid, which bears the elastic property. For the details on this model see [7].

The local existence and uniqueness of the Oldroy model on entire space $\mathbb{R}^n$ or a periodic domain was established by Lin etc. in [7], where the global existence and uniqueness of smooth solution with small initial data was also established see also [5]. The wellposedness on a bounded smooth domain with Dirichlet conditions was established by Lin and Zhang in [8].

We remark some properties of the deformation tensor. Let x be the Euler coordinate and X the Lagrangian coordinate. For a given velocity field $u(t,x)$ the flow map $x(t,X)$ is defined by the following ordinary differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} x(t,X) = u(t, x(t,X)), \\ x(0,X) = X. \end{cases}$$

The deformation tensor is $\tilde{F}(t,X) = \frac{\partial x}{\partial X}(t,X)$. In the Eulerian coordinate, the corresponding deformation tensor is define as $F(t, x(t,X)) = \tilde{F}(t,X)$. Differentiating its both sides with respect to t by chain rule one obtain the second equation of (1.1), which says that $\partial_t F_{ij} + u_k\cdot\partial_{x_k} F_{ij} = \partial_{x_k} u_i F_{kj}$ for $i,\ j = 1,2,\cdots,n$, in the $(i,j)-$th entries, where we use the Einstein summation convention that the repetition index denotes sum over 1 to n.

If div $F(0,x) = 0$, then from the second equation of Oldroy (1.1) we have

$$\partial_t(\nabla\cdot F^t) + u\cdot\nabla(\nabla\cdot F^t) = 0. \tag{1.2}$$

Therefore, $\nabla\cdot F^t = 0$ for any $t > 0$.

Denote the ith column of F as $F_{\cdot i}$, then $\nabla\cdot(FF^t) = F_{\cdot i}\cdot\nabla F_{\cdot i}$ by the fact $\nabla\cdot F^t = 0$. So the system (1.1) can be rewritten in an equivalent form

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = F_{\cdot i}\cdot\nabla F_{\cdot i}, \\ \partial_t F_{\cdot k} + u\cdot\nabla F_{\cdot k} = F_{\cdot k}\cdot\nabla u,\ k = 1,\cdots,n, \\ \mathrm{div} u = 0,\ \mathrm{div} F = 0. \end{cases} \tag{1.3}$$

In reference [7], Lin, Liu and Zhang obtained the local existence and uniqueness of smooth solution for smooth initial data, and had a blowup criterion.

Theorem (Lin, Liu and Zhang) For smooth initial data $(u_0, F_0) \in H^2(\mathbb{R}^n)$, there exists a positive time $T = T(\|u_0\|_{H^2}, \|F_0\|_{H^2})$ such that the system (1.1) possesses a unique smooth solution on $[0,T]$ with

$$(u,F) \in L^\infty([0,T]; H^2(\mathbb{R}^n)) \cap L^2([0,T]; H^3(\mathbb{R}^n)).$$

Moreover, if T^* is the maximal time of existence, then

$$\int_0^{T^*} \|\nabla u\|_{H^2}^2 \mathrm{d}s = +\infty.$$

In reference [3], Hu and Hynd study the blowup criterion for the ideal viscoelastic flow, which is the Oldroy system (1.1) in the case of $\mu = 0$. They showed an Beale-Kato-Majda [1] type blowup criterion that the smooth solution to the Oldroy flow do not develop singularity for $t \le T$ provided that

$$\int_0^T \|\nabla\times u\|_{L^\infty(\mathbb{R}^3)}\mathrm{d}s + \sum_{k=1}^3 \int_0^T \|\nabla\times F_{\cdot k}\|_{L^\infty(\mathbb{R}^3)}\mathrm{d}s < +\infty.$$

From the modeling of Oldroy system we know that the deformation tensor can be determined by the velocity u of the flow. Therefore we consider the blowup criterion of smooth solution by means of only $\|\nabla u\|_\infty$. In fact, Zhao, Guo and Huang [12] constructed a set of finite time blowup solution in two dimension case:

$$u(t,x) = \left(\frac{x_1 f_0}{1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t}, \frac{x_2 f_0}{1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t}\right)^t,\ p(t,x) = \frac{(\alpha x_1^2 - \beta x_2^2) f_0^2}{(\beta-\alpha)(1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t)^2},$$
$$F(t,x) = \mathrm{diag}\left(\left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\beta-\alpha}{\alpha+\beta}}, \left|1 - \frac{\alpha+\beta}{\alpha-\beta} f_0 t\right|^{\frac{\beta+\alpha}{\alpha-\beta}}\right).$$

If $\frac{\alpha+\beta}{\alpha-\beta} f_0 > 0$, $\alpha+\beta \ne 0$ and $\alpha-\beta \ne 0$, then the above solution will blow up at time $T^* = \frac{\alpha-\beta}{(\alpha+\beta)f_0}$. We see that

$$\int_0^{T^*} \|\nabla u(t)\|_\infty \mathrm{d}t = +\infty.$$

There are other types of blowup criteria of smooth solutions to the Oldroy models, for example [6, 2]. To this end, we state our main results.

Theorem 1.1. *Let $u_0 \in H^2(\mathbb{R}^n)$ and $F_0 \in H^2(\mathbb{R}^n)$ with $\nabla\cdot u_0 = \nabla\cdot F_{\cdot k,0} = 0$ for $k = 1,\cdots,n$. Assume the pair $(u,F) \in L^\infty([0,T]; H^2(\mathbb{R}^n)) \cap L^2([0,T]; H^3(\mathbb{R}^n))$ is a smooth solution to the Oldroy system (1.3). Then the smooth solution do not appear breakdown until $T^* > T$ provided that*

$$\int_0^{T^*} \|\nabla u(t)\|_\infty dt < +\infty. \tag{1.4}$$

Remark 1.1. *For the local smooth solution $(u,F) \in L^\infty([0,T]; H^2(\mathbb{R}^n)) \cap L^2([0,T]; H^3(\mathbb{R}^n))$, if T^* is its maximum existence time, then $\int_0^{T^*} \|\nabla u(t)\|_\infty dt = +\infty$.*

In the second section we will prove the Theorem 1.1 for the case $n = 2$, which can be done by energy estimates. The L^2 and H^1 energy estimates are the same for the case $n = 2$ and $n = 3$. In the H^2 energy estimate, we use the Sobolev interpolation inequality $\|\nabla F\|_4^2 \leq C\|\nabla F\|_2\|\Delta F\|_2$. In case $n = 3$, however, the inequality is $\|\nabla F\|_4^2 \leq C\|\nabla F\|_2^{\frac{1}{2}}\|\Delta F\|_2^{\frac{3}{2}}$ which does not match the H^2 energy estimate, because it will result in the appearance of the term $\|\Delta F\|_2^3$ that the power is higher that the left hand side. We obtain the H^2 energy estimate of u by virtue of the momentum equation, combining the H^2 estimate of u and F again with the estimate of $\|\nabla F\|_{L^6}$ we grasp the H^2 energy estimate of u and F finally. The section three will devote to the proof of the case $n = 3$.

In this paper C denote a harmless constant which may be dependent on dimension n, the norm of initial data, the viscosity μ, but not dependent on the estimated quantity. We denote the L^p norm of a function f by $\|f\|_p$ or $\|f\|_{L^p}$. We denote the derivative with respect to x_i by ∂_i or ∂_{x_i}. We also use f_t to denote the derivative of f with respect to t.

2. Proof of the case $n = 2$

(1) L^2-energy estimate and L^p estimate of the deformation tensor F

The L^2-energy estimate can be easily obtained by the standard L^2 inner product process.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_2^2 + \|F_{\cdot k}\|_2^2) + \mu\|\nabla u\|_2^2 = (F_{\cdot i} \cdot \nabla F_{\cdot i}, u) + (F_{\cdot k} \cdot \nabla u, F_{\cdot i}) = 0.$$

So we have

$$\|u\|_2^2 + \|F\|_2^2 + 2\mu \int_0^t \|\nabla u\|_2^2 \mathrm{d}s = \|u_0\|_2^2 + \|F(0)\|_2^2. \tag{2.1}$$

Multiplying both sides of the second equation of (1.3) by $p|F_{\cdot k}|^{p-2}F_{\cdot k}$ for $2 \leq p < \infty$ and integrating both sides on $\mathbb{R}^n$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|F_{\cdot k}\|_p^p \leq p\|\nabla u\|_\infty\|F\|_p^p. \tag{2.2}$$

Summing up the estimate (2.2) with respect to k one has

$$\|F\|_p \leq \|F_0\|_p \exp\left\{C(n)\int_0^t \|\nabla u(s)\|_\infty \mathrm{d}s\right\}. \tag{2.3}$$

Let $p \to \infty$, we have

$$\|F\|_\infty \leq \|F_0\|_\infty \exp\left\{C(n)\int_0^t \|\nabla u(s)\|_\infty \mathrm{d}s\right\}. \tag{2.4}$$

(2) $\dot{H}^1$-energy estimate

We differentiate the equations (1.3) with respect to x_i, then multiply the resulting equations by $\partial_i u$ and $\partial_i F_{\cdot j}$ for $i = 1,\ 2$, integrate with respect to x and sum them up. It follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_i u\|_2^2 + \|\partial_i F\|_2^2) + \mu\|\partial_i \nabla u\|_2^2 \leq \\
&|(\partial_i u \cdot \nabla u, \partial_i u)| + |(\partial_i F_{\cdot k} \cdot \nabla F_{\cdot k}, \partial_i u)| + |(\partial_i u \cdot \nabla F_{\cdot j}, \partial_i F_{\cdot j})| + |(\partial_i F_{\cdot j} \cdot \nabla u, \partial_i F_{\cdot j})|,
\end{aligned}$$

where use has been made of the facts

$$\begin{aligned}
&(u \cdot \nabla \partial_i u, \partial_i u) = (u \cdot \partial_i F_{\cdot j}, \partial_i F_{\cdot j}) = (\nabla \partial_i p, \partial_i u) = 0,\\
&(F_{\cdot k} \cdot \nabla \partial_i F_{\cdot k}, \partial_i u) + (F_{\cdot j} \cdot \nabla \partial_i u, \partial_i F_{\cdot j}) = 0.
\end{aligned}$$

Noting that

$$|(\partial_i u \cdot \nabla u, \partial_i u)| \leq \|\nabla u\|_\infty \|\nabla u\|_2^2,$$

$$|(\partial_i F_{\cdot k} \cdot \nabla F_{\cdot k}, \partial_i u)|,\ |(\partial_i u \cdot \nabla F_{\cdot j}, \partial_i F_{\cdot j})|,\ |(\partial_i F_{\cdot j} \cdot \nabla u, \partial_i F_{\cdot j})| \leq \|\nabla u\|_\infty \|\nabla F\|_2^2.$$

So
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\nabla u\|_2^2+\|\nabla F\|_2^2)+\mu\|D^2u\|_2^2\le C\|\nabla u\|_\infty(\|\nabla u\|_2^2+\|\nabla F\|_2^2).$$
Gronwall's inequality implies
$$\|\nabla u\|_2^2+\|\nabla F\|_2^2+2\mu\int_0^t\|D^2u\|_2^2\mathrm{d}s\le(\|\nabla u_0\|_2^2+\|\nabla F(0)\|_2^2)\exp\left\{\int_0^t C\|\nabla u(s)\|_\infty\mathrm{d}s\right\}. \tag{2.5}$$

(3) $\dot{H}^2$-energy estimate

Applying operator Δ on both sides of (1.3), we have
$$\begin{cases}\partial_t\Delta u-\mu\Delta^2u+\Delta u\cdot\nabla u+u\cdot\nabla\Delta u+2\partial_i u\cdot\nabla\partial_i u+\nabla\Delta p=\Delta F_{\cdot k}\cdot\nabla F_{\cdot k}+F_{\cdot k}\nabla\Delta F_{\cdot k}+2\partial_i F_{\cdot k}\cdot\nabla\partial_i F_{\cdot k}\\ \partial_t\Delta F_{\cdot k}+\Delta u\cdot\nabla F_{\cdot k}+u\cdot\nabla\Delta F_{\cdot k}+2\partial_i u\cdot\nabla\partial_i F_{\cdot k}=\Delta F_{\cdot k}\cdot\nabla u+F_{\cdot k}\cdot\nabla\Delta u+2\partial_i F_{\cdot k}\cdot\nabla\partial_i u.\end{cases} \tag{2.6}$$

Taking the L^2 inner of equation (2.6) with Δu and $\Delta F_{\cdot k}$ and summing them up, one can obtain that
$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_2^2+\|\Delta F\|_2^2)+\mu\|\Delta\nabla u\|_2^2\\ &\le|(\Delta u\cdot\nabla u,\Delta u)|+2|(\partial_i u\cdot\nabla\partial_i u,\Delta u)|+|(\Delta F_{\cdot k}\cdot\nabla F_{\cdot k},\Delta u)|\\ &+2|(\partial_i F_{\cdot k}\cdot\nabla\partial_i F_{\cdot k},\Delta u)|+|(\Delta u\cdot\nabla F_{\cdot k},\Delta F_{\cdot k})|+2|(\partial_i u\cdot\nabla\partial_i F_{\cdot k},\Delta F_{\cdot k})|\\ &+|(\Delta F_{\cdot k}\cdot\nabla u,\Delta F_{\cdot k})|+2|(\partial_i F_{\cdot k}\cdot\nabla\partial_i u,\Delta F_{\cdot k})|.\end{aligned} \tag{2.7}$$

Here use has been made of the the facts that
$$(u\cdot\nabla\Delta u,\Delta u)=0,\ \ (u\cdot\nabla\Delta F_{\cdot k},\Delta F_{\cdot k})=0,$$
$$(F_{\cdot k}\cdot\nabla\Delta F_{\cdot k},\Delta u)+(F_{\cdot k}\cdot\nabla\Delta u,\Delta F_{\cdot k})=0.$$

Noting that
$$|(\Delta u\cdot\nabla u,\Delta u)|,\ \ |(\partial_i u\cdot\nabla\partial_i u,\Delta u)|\le C\|\nabla u\|_\infty\|\Delta u\|_2^2,$$
$$|(\Delta F_{\cdot k}\cdot\nabla u,\Delta F_{\cdot k})|,\ \ |(\partial_i u\cdot\nabla\partial_i F_{\cdot k},\Delta F_{\cdot k})|\le C\|\nabla u\|_\infty\|\Delta F\|_2^2.$$
$$\begin{aligned}&|(\Delta F_{\cdot k}\cdot\nabla F_{\cdot k},\Delta u)+2(\partial_i F_{\cdot k}\cdot\nabla\partial_i F_{\cdot k},\Delta u)|\\ &=|-(\partial_i F_{\cdot k}\cdot\nabla F_{\cdot k},\partial_i\Delta u)-(\partial_i F_{\cdot k}\cdot\nabla\Delta u,\partial_i F_{\cdot k})|\le C\|\nabla\Delta u\|_2\|\nabla F\|_4^2\\ &\le\frac{\mu}{4}\|\nabla\Delta u\|_2^2+C\|\nabla F\|_2^2\|\Delta F\|_2^2,\end{aligned}$$
where we have used the Sobolev interpolation inequality
$$\|\nabla F\|_4^2\le C\|\nabla F\|_2\|\Delta F\|_2.$$

Arguing similarly as the above, one has
$$\begin{aligned}&|(\Delta u\cdot\nabla F_{\cdot k},\Delta F_{\cdot k})|=|(\partial_i\Delta u\cdot\nabla F_{\cdot k},\partial_i F_{\cdot k})|\le\frac{\mu}{8}\|D^3u\|_2^2+C\|\nabla F\|_2^2\|\Delta F\|_2^2,\\ &|(\partial_i F_{\cdot k}\cdot\nabla\partial_i u,\Delta F_{\cdot k})|=|(\partial_i\partial_j F_{\cdot k}\cdot\nabla\partial_j F_{\cdot k},\partial_i u)|+|(\partial_i F_{\cdot k}\cdot\nabla\partial_i\partial_j u,\partial_j F_{\cdot k})|\\ &\le C\|\nabla u\|_\infty\|\Delta F\|_2^2+\frac{\mu}{8}\|\nabla\Delta u\|_2^2+C\|\nabla F\|_2^2\|\Delta F\|_2^2.\end{aligned}$$

Inserting the above estimates into estimate (2.7), it can be derived that
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_2^2+\|\Delta F\|_2^2)+\frac{\mu}{2}\|\Delta\nabla u\|_2^2\le C\|\nabla u\|_\infty^2(\|\Delta u\|_2^2+\|\Delta F\|_2^2)+C\|\nabla F\|_2^2\|\Delta F\|_2^2.$$

Gronwall's inequality implies that
$$\|\Delta u\|_2^2+\|\Delta F\|_2^2+\mu\int_0^t\|\Delta\nabla u(s)\|_2^2\mathrm{d}s\le(\|\Delta u_0\|_2^2+\|\Delta F(0)\|_2^2)\exp\left\{C\exp\int_0^t Ct\|\nabla u\|_\infty\mathrm{d}s\right\}. \tag{2.8}$$

(4) Higher derivative estimates.

Next we derive the higher derivative estimate of u and F. For this purpose we need the following commutator estimate.

Proposition 2.1. (Kato and Ponce [4], [9]) *Let $1 < p < \infty$ and $0 < s$. Assume that $f,\ g \in W^{s,p}$, then there exists a abstract constant C such that*

$$\|[J^s, f]g\|_p \le C(\|\nabla f\|_{p_1}\|g\|_{W^{s-1,p_2}} + \|f\|_{W^{s,p_3}}\|g\|_{p_4}) \tag{2.9}$$

$$\|[\Lambda^s, f]g\|_p \le C(\|\nabla f\|_{p_1}\|\Lambda^{s-1}g\|_{p_2} + \|\Lambda^s f\|_{p_3}\|g\|_{p_4}) \tag{2.10}$$

with $1 < p_2, p_3 < \infty$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

where $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$ and $\Lambda^s = (-\Delta)^{\frac{1}{2}},\ J = (1-\Delta)^{1/2}$.

Applying Λ^s on both sides of (1.3) and taking the inner product with $\Lambda^s u$ and $\Lambda^s F$, it can be derived that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_2^2 + \|\Lambda^s F_{\cdot k}\|_2^2) + \mu\|\Lambda^{s+1}u\|_2^2 \le \\ &|(\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u, \Lambda^s u)| + |(\Lambda^s(F_{\cdot k}\cdot\nabla F_{\cdot k}) - F_{\cdot k}\cdot\nabla\Lambda^s F_{\cdot k}, \Lambda^s u)| + \\ &|(\Lambda^s(u\cdot\nabla F_{\cdot k}) - u\cdot\nabla\Lambda^s F_{\cdot k}, \Lambda^s F_{\cdot k})| + |(\Lambda^s(F_{\cdot k}\cdot\nabla u) - F_{\cdot k}\cdot\nabla\Lambda^s u, \Lambda^s F_{\cdot k})|,\end{aligned} \tag{2.11}$$

where we have used the facts

$$(F_{\cdot k}\cdot\nabla\Lambda^s F_{\cdot k}, \Lambda^s u) + (F_{\cdot k}\cdot\nabla\Lambda^s u, \Lambda^s F_{\cdot k}) = 0,$$

$$(u\cdot\nabla\Lambda^s F_{\cdot k}, \Lambda^s F_{\cdot k}) = (u\cdot\nabla\Lambda^s u, \Lambda^s u) = 0.$$

The commutator estimate (2.10) implies that

$$\|\Lambda^s(u\cdot\nabla u) - u\cdot\nabla\Lambda^s u\|_2 \le \|\nabla u\|_\infty\|\Lambda^s u\|_2,$$

$$\|\Lambda^s(F_{\cdot k}\cdot\nabla F_{\cdot k}) - F_{\cdot k}\cdot\nabla\Lambda^s F_{\cdot k}\|_2 \le \|\nabla F\|_\infty\|\Lambda^s F\|_2 \le \|\nabla F\|_{H^{s-1}}\|\Lambda^s F\|_2,$$

$$\|\Lambda^s(u\cdot\nabla F_{\cdot k}) - u\nabla\Lambda^s F_{\cdot k}\| \le \|\nabla u\|_\infty\|\Lambda^s F\|_2 + \|F\|_\infty\|\Lambda^{s+1}u\|_2,$$

$$\|\Lambda^s(F_{\cdot k}\cdot\nabla u) - F_{\cdot k}\nabla\Lambda^s u\| \le \|\nabla u\|_\infty\|\Lambda^s F\|_2 + \|F\|_\infty\|\Lambda^{s+1}u\|_2,$$

where the Sobolev embedding $H^{s-1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ for $s > 1 + \frac{n}{2}$ is applied.

Inserting the above estimates into estimate (2.11), it follows

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\Lambda^s u\|_2^2 + \|\Lambda^s F_{\cdot k}\|_2^2) + \frac{\mu}{2}\|\Lambda^{s+1}u\|_2^2 \le \\ &C(\|\nabla u\|_\infty + \|\nabla F\|_2 + \|\Lambda^s u\|_2 + \|F\|_\infty^2)(\|\Lambda^s u\|_2^2 + \|\Lambda^s F\|_2^2),\end{aligned} \tag{2.12}$$

where we have used the fact

$$\|\nabla F\|_{H^{s-1}}\|\Lambda^s F\|_2\|\Lambda^s u\|_2 \le \|\nabla F\|_2(\|\Lambda^s F\|_2^2 + \|\Lambda^s u\|_2^2) + \|\Lambda^s F\|_2^2\|\Lambda^s u\|_2.$$

So, for $s \ge 3$, applying Gronwall's inequality to (2.12), by induction for u's estimate, we obtain the higher derivative estimate:

$$\begin{aligned}&\|\Lambda^s u\|_2^2 + \|\Lambda^s F\|_2^2 + \mu\int_0^t \|\Lambda^{s+1}u\|_2^2 \mathrm{d}s \le \\ &(\|u_0\|_{H^s}^2 + \|F(0)\|_{H^s}^2)\exp\left\{\int_0^t C(\|\nabla u\|_\infty + \|\nabla F\|_2 + \|\Lambda^s u\|_2 + \|F\|_\infty^2)\mathrm{d}s\right\}.\end{aligned}$$

Therefore, we complete the proof of the case $n = 2$.

3. Proof of the case $n = 3$

In the three dimensional case the L^2 and H^1 energy estimates are the same as the case of dimension two. To estimate the H^2 energy estimate we need the following estimates.

Multiplying the first equation of (1.3) by u_t and integrating both sides over $\mathbb{R}^3$ with respect to x, and noting div $u = 0$, it follows

$$\begin{aligned}\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2 + \|u_t\|_2^2 &\le |(u\cdot\nabla u, u_t)| + |(F_{\cdot k}\cdot\nabla F_{\cdot k}, u_t)| \\ &\le \frac{1}{2}\|u_t\|_2^2 + C\|u\|_\infty^2\|\nabla u\|_2^2 + C\|\nabla F\|_2^2\|F\|_\infty^2.\end{aligned}$$

Integrating both sides with respect to t it yields

$$\mu\|\nabla u\|_2^2 + \int_0^t \|u_t\|_2^2\mathrm{d}s \le \mu\|\nabla u_0\|_2^2 + \sup_{0<s<t}\|\nabla u\|_2^2\int_0^t\|u\|_{H^2}^2\mathrm{d}s + \int_0^t\|F\|_\infty^2\|\nabla F\|_2^2\mathrm{d}s \tag{3.1}$$

where the Sobolev embedding $H^2(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ has been used.

Differentiating the first equation of (1.3) with respect to t, we arrive at

$$u_{tt} - \mu\Delta u_t + u_t\cdot\nabla u + u\cdot\nabla u_t + \nabla p_t = F_{\cdot kt}\cdot\nabla F_{\cdot k} + F_{\cdot k}\cdot\nabla F_{\cdot kt}. \tag{3.2}$$

Taking L^2 inner product of the equation (3.2) with respect to u_t, it can be similarly derived that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_t\|_2^2 + \mu\|\nabla u_t\|_2^2 &\le \|\nabla u\|_\infty\|u_t\|_2^2 + 2\|F\|_\infty\|\nabla u_t\|_2\|F_t\|_2 \\ &\le \frac{\mu}{2}\|\nabla u_t\|_2^2 + \|\nabla u\|_\infty\|u_t\|_2^2 + C\|F\|_\infty^2\|F_t\|_2^2.\end{aligned}$$

Applying the Gronwall' inequality, it yields

$$\|u_t\|_2^2 + \mu\int_0^t\|\nabla u_t\|_2^2\mathrm{d}s \le (\|u_t(0)\|_2^2 + C\int_0^t\|F\|_\infty^2\|F_t\|_2^2\mathrm{d}s)\exp\Big\{\int_0^t\|\nabla u\|_\infty\mathrm{d}s\Big\}. \tag{3.3}$$

It need still to estimate $\|F_t\|_2^2$. From the second equation of (1.3) it can be derived that

$$\begin{aligned}\|F_t\|_2^2 &\le \|F_t\|_2\|u\|_\infty\|\nabla F\|_2 + \|F_t\|_2\|F\|_\infty\|\nabla u\|_2 \\ &\le \frac{1}{2}\|F_t\|_2^2 + C\|u\|_\infty^2\|\nabla F\|_2^2 + C\|F\|_\infty^2\|\nabla u\|_2^2.\end{aligned}$$

So we arrive at

$$\|F_t\|_2^2 \le C\|u\|_\infty^2\|\nabla F\|_2^2 + C\|F\|_\infty^2\|\nabla u\|_2^2.$$

Inserting it to the estimate (3.3) we obtain the estimate of $\|u_t\|_2$:

$$\|u_t\|_2^2 + \mu\int_0^t\|\nabla u_t\|_2^2\mathrm{d}s \le C(t) < \infty, \tag{3.4}$$

where $C(t)$ is explicit increasing function of t dependent on $\int_0^t\|\nabla u\|_\infty\mathrm{d}s$. From the first equation of (1.3), ∇p can be solved by Riesz transformation $R = (R_1, R_2, R_3)^t$, with $R_j = -i\partial_{x_j}(-\Delta)^{-\frac{1}{2}}$ being the jth Riesz transformation.

$$\nabla p = RR\cdot(u\cdot\nabla u) - RR\cdot(F_{\cdot k}\cdot\nabla F_{\cdot k}).$$

In virtue of the boundedness of Riesz operator R in L^p space for $1 < p < \infty$, we obtain that

$$\|\nabla p\|_2 \le C\|\nabla u\|_2\|u\|_\infty + C\|\nabla F\|_2\|F\|_\infty.$$

For details about Riesz transformation see [10, 11].

Thus from the first equation of (1.3) we have

$$\begin{aligned}\mu\|\Delta u\|_2 &\le \|u_t\|_2 + \|u\cdot\nabla u\|_2 + \|\nabla p\|_2 + \|F_{\cdot k}\cdot\nabla F_{\cdot k}\|_2 \\ &\le \|u_t\|_2 + \frac{\mu}{2}\|\Delta u\|_2 + C\|u\|_2\|\nabla u\|_2^4 + C\|F\|_\infty\|\nabla F\|_2,\end{aligned}$$

where the interpolation inequality $\|u\|_\infty \le C\|u\|_2^{\frac14}\|\Delta u\|_2^{\frac34}$ has been used. So we derive

$$\|\Delta u\|_2 \le C(\|u_t\|_2 + \|u\|_2\|\nabla u\|_2^4 + \|F\|_\infty\|\nabla F\|_2). \tag{3.5}$$

Next we derive the estimate of $\|\Delta F\|_2$. Applying Δ on the both sides of equation (1.3) and taking the L^2 inner product with Δu and $\Delta F_{\cdot k}$ respectively, we have

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|_2^2 + \mu\|\Delta\nabla u\|_2^2 \le |(\Delta(u\cdot\nabla u) - u\cdot\nabla\Delta u, \Delta u)| + |(\Delta(F_{\cdot k}\cdot\nabla F_{\cdot k}) - F_{\cdot k}\cdot\nabla\Delta F_{\cdot k}, \Delta u)|, \tag{3.6}$$

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta F_{\cdot k}\|_2^2 \le |(\Delta(u\cdot\nabla F_{\cdot k}) - u\cdot\nabla\Delta F_{\cdot k}, \Delta F_{\cdot k})| + |(\Delta(F_{\cdot k}\cdot\nabla u) - F_{\cdot k}\cdot\nabla\Delta u, \Delta F_{\cdot k})|, \tag{3.7}$$

where use has been of the facts

$$(u\cdot\nabla\Delta u, \Delta u) = (u\cdot\nabla\Delta F_{\cdot k}, \Delta F_{\cdot k}) = 0,$$
$$(F_{\cdot k}\cdot\nabla\Delta F_{\cdot k}, \Delta u) + (F_{\cdot k}\cdot\nabla\Delta u, \Delta F_{\cdot k}) = 0.$$

Next we estimate the right hand sides. By the communicator estimate (2.10) one has

$$|(\Delta(u\cdot\nabla u) - u\cdot\nabla\Delta u, \Delta u)| \le \|\Delta u\|_2\|\Delta(u\cdot\nabla u) - u\cdot\nabla\Delta u\|_2 \le \|\nabla u\|_\infty\|\Delta u\|_2^2,$$
$$\begin{aligned}|(\Delta(u\cdot\nabla F_{\cdot k}) - u\cdot\nabla\Delta F_{\cdot k}, \Delta F_{\cdot k})| &\le \|\Delta F\|_2(\|\nabla u\|_\infty\|\Delta F\|_2 + \|F\|_\infty\|\nabla\Delta u\|_2)\\ &\le \|\nabla u\|_\infty\|\Delta F\|_2^2 + C\|F\|_\infty^2\|\Delta F\|_2^2 + \frac{\mu}{8}\|\nabla\Delta u\|_2^2,\end{aligned}$$
$$|(\Delta(F_{\cdot k}\cdot\nabla u) - F_{\cdot k}\cdot\nabla\Delta u, \Delta F_{\cdot k})| \le \|\nabla u\|_\infty\|\Delta F\|_2^2 + C\|F\|_\infty^2\|\Delta F\|_2^2 + \frac{\mu}{8}\|\nabla\Delta u\|_2^2.$$

For the second term on the right hand side of (3.6) we estimate as follows

$$|(\Delta(F_{\cdot k}\cdot\nabla F_{\cdot k}) - F_{\cdot k}\cdot\nabla\Delta F_{\cdot k}, \Delta u)| \le \|\Delta u\|_6\|\Delta(F_{\cdot k}\cdot\nabla F_{\cdot k}) - F_{\cdot k}\cdot\nabla\Delta F_{\cdot k}\|_{6/5},$$

and

$$\|\Delta(F_{\cdot k}\cdot\nabla F_{\cdot k}) - F_{\cdot k}\cdot\nabla\Delta F_{\cdot k}\|_{6/5} \le \|\nabla F\|_6\|\Delta F\|_{3/2} \le \|\nabla F\|_6\|\nabla F\|_2^{\frac12}\|\Delta F\|_2^{\frac12}.$$

So one has the estimate

$$|(\Delta(F_{\cdot k}\cdot\nabla F_{\cdot k}) - F_{\cdot k}\cdot\nabla\Delta F_{\cdot k}, \Delta u)| \le \frac{\mu}{4}\|\nabla\Delta u\|_2^2 + C\|\nabla F\|_6^4 + C\|\nabla F\|_2^2\|\Delta F\|_2^2.$$

Summing up (3.6) and (3.7), and inserting the above estimates into the summation, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_2^2 + \|\Delta F_{\cdot k}\|_2^2) + \mu\|\nabla\Delta u\|_2^2 \le C(\|\nabla u\|_\infty + \|F\|_\infty^2 + \|\nabla F\|_2^2)(\|\Delta u\|_2^2 + \|\Delta F\|_2^2) + C\|\nabla F\|_6^4. \tag{3.8}$$

We still have to estimate $\|\nabla F\|_6$. Differentiating the second equation of (1.3) with respect to x_i, one has

$$\partial_t\partial_i F_{\cdot k} + \partial_i u\cdot\nabla F_{\cdot k} + u\cdot\nabla\partial_i F_{\cdot k} = \partial_i F_{\cdot k}\cdot\nabla u + F_{\cdot k}\cdot\nabla\partial_i u.$$

Multiplying both sides of the above equation by $6|\partial_i F_{\cdot k}|^4\partial_i F_{\cdot k}$, and integrating both sides with respect to x over $\mathbb{R}^3$, it can be derived that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla F\|_6^4 \le C\|\nabla u\|_\infty\|\nabla F\|_6^4 + C\|F\|_\infty\|\Delta u\|_6\|\nabla F\|_6^3. \tag{3.9}$$

Next we have to derive an estimate of $\|\Delta u\|_6$. Using an argument similar to deriving the L^2 estimate $\|\Delta u\|_2$ in (3.5) we have

$$\begin{aligned}\mu\|\Delta u\|_6 &\le \|\partial_t u\|_6 + \|u\|_\infty\|\nabla u\|_6 + C\|F\|_\infty\|\nabla F\|_6\\ &\le \|\partial_t\nabla u\|_2 + C\|u\|_\infty\|\Delta u\|_2 + C\|F\|_\infty\|\nabla F\|_6.\end{aligned} \tag{3.10}$$

Inserting estimates (3.10) to (3.9) one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla F\|_6^4 &\le C(\|\nabla u\|_\infty + \|F\|_\infty^2 + \|\partial_t\nabla u\|_2^2)\|\nabla F\|_6^4 + C\|F\|_\infty\|u\|_\infty\|\Delta u\|_2\|\nabla F\|_6^3 + C\\ &\le C(\|\nabla u\|_\infty + \|F\|_\infty^2 + \|\partial_t\nabla u\|_2^2 + 1)\|\nabla F\|_6^4 + C\|F\|_\infty^4\|u\|_2\|\Delta u\|_2^7 + C.\end{aligned} \tag{3.11}$$

Combining the estimates (3.8) and (3.11) we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\Delta u\|_2^2+\|\Delta F_{\cdot k}\|_2^2+\|F\|_6^4)+\mu\|\nabla\Delta u\|_2^2\leq$$
$$C(\|\nabla u\|_\infty+\|\nabla F\|_2^2+\|F\|_\infty^2+\|\partial_t\nabla u\|_2^2+1)(\|\Delta u\|_2^2+\|\Delta F\|_2^2+\|F\|_6^4)+C\|F\|_\infty^4\|u\|_2\|\Delta u\|_2^7+C.$$

Gronwall's inequality implies the H^2 estimates:

$$\begin{aligned}&\|\Delta u\|_2^2+\|\Delta F_{\cdot k}\|_2^2+\|F\|_6^4+\mu\int_0^t\|\nabla\Delta u\|_2^2\mathrm{d}s\leq\exp\left\{C(t)\int_0^t(\|\nabla u\|_\infty+\|\partial_t\nabla u\|_2^2)\mathrm{d}s\right\}\times\\&\left(\|\Delta u(0)\|_2^2+\|\Delta F_{\cdot k}(0)\|_2^2+\|F(0)\|_6^4+C\int_0^t(\|F\|_\infty^4\|u\|_2\|\Delta u\|_2^7+1)\mathrm{d}s\right)<\infty.\end{aligned}\tag{3.12}$$

Based on the H^2 energy estimate the higher energy estimate can be obtained by bootstrap method as we did in section two. Thus the proof of the case $n=3$ is completed.

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References

1. J.T. Beal, T. Kato and A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations, *Comm. Math. Phys.*, **94** (1984), 61-66.
2. J.Y. Chemin, N. Masmoudi, About lifespan of regular solutions 0f equations related to viscoelastic fluid, *SIAM J. Math. Anal.*, **33**(2001), 84-112.
3. X. P. Hu, R. Hynd, A blowup criterion for ideal viscelastic flow, *arXiv*: 1102.1113v1 [math.AP] 6 Feb 2011.
4. T. Kato, G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, *comm Pure Appl. Math.*, **41** (1988), 891-907.
5. Z. Lei, C. Liu, Y. Zhou, Global solutions for incompressible viscoelastic fluids, *Arch. Rational Mech. Anal.*, **188** (2008), 371-398.
6. Z. Lei, N. Masmoudi, Y. Zhou, Remarks on the blowup croteria for Oldroyd models, *J. differential Equations*, **248** (2010), 328-341.
7. F. H. Lin, C. Liu, P. Zhang, On hydrodynamics of viscoelastic fluids, *Comm. Pure APPl. Math.*, **Vol. LVIII**(2005), 1437-1471.
8. F. H. Lin, P. Zhang, On the initial-boundary value problem of the incompressible viscoelastic fluid system, *Comm. Pure APPl. Math.*, **Vol. LVI** (2008), 539-558.
9. A. J. Majda, A. L. Bertozzi, Vorticity and Incompressible Flow, Cambridge Univ. Press, 2002.
10. Changxing Miao, Harmonic analysis and application to partial differential equations, Science Press, Beijing, 2004, second edition.
11. E. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces. Princeton Univ. Press, 1971.
12. L. G. Zhao, B. L. Guo, H. Y. Huang, Blow-up solutions to a viscoelastic fluid system and a coupled Navier-Stokes/phase-field system in $\mathbb{R}^2$.

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